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NATURAL FREQUENCY AND MODE SHAPE SENSITIVITIES OF DAMPED SYSTEMS: PART II, MULTIPLE NATURAL FREQUENCIES

I.-W. LEE, D.-O. KIM

Department of Civil Engineering, Korea Advanced Institute of Science and Technology, Science Town, Taejon 305-701, *Korea*

AND

G.-H. JUNG

Department of Mechanical Engineering, Korea Advanced Institute of Science and Technology, Science Town, Taejon 305-701, *Korea*

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An efficient algorithm with proven numerical stability is derived for computation of eigenvalue and eigenvector derivatives of damped vibratory systems with multiple eigenvalues. In the proposed method, adjacent eigenvectors and orthonormal conditions are used to compose an algebraic equation whose order is $(n+m)\times(n+m)$, where n is the number of co-ordinates and m the number of multiplicity of a multiple natural frequency. The mode shape derivatives of the damped systems can be obtained by solving the algebraic equation. The method can be consistently applied to both structural systems with structural design parameters and mechanical systems with lumped design parameters. As an example of a structural system to demonstrate the theory of the proposed method and its possibilities in the case of multiple eigenvalues, the finite element model of the cantilever beam is considered, and also a 5-DOF mechanical system in the case of a non-proportionally damped system. The design parameter of the cantilever beam is its height, and that of the 5-DOF mechanical system is a spring.

1. INTRODUCTION

The eigenpair sensitivities of structural and mechanical systems with multiple natural frequencies have been a focus of recent interest. In typical structural or mechanical systems, there are many multiple or nearly equal natural frequencies, due to their structural symmetries or certain reasons. In this case, since eigenspace spanned by the mode shapes corresponding to the multiple natural frequencies is degenerate, any linear combination of mode shapes can be a mode shape. A number of papers [1–8] have been presented to find the mode shape derivatives in the case of multiple natural frequencies. For the mode shape derivative to be found, the adjacent mode shapes which lie "adjacent" to the m

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(multiplicity of multiple natural frequency) distinct mode shapes appearing when a design parameter varies must be calculated first. To do so, the approximate mode shapes could be varied continuously by varying the design parameter. For the real symmetric case, a generalization of Nelson's method [9] was obtained by Ojalvo [1] and amended by Mills-Curren [2] and Dailey [3]. Dailey's method is an exact analytical method for calculating mode shape derivatives. This method only requires knowledge of the eigenpair with multiple eigenvalues, however, the method is lengthy and complicated for finding mode shape derivatives and clumsy for programming. Dailey's method is extremely complicated for calculating the sensitivity of eigenvectors of multiple eigenvalues in the case of the damped systems.

In this paper the algebraic method for calculating the natural frequencies and the corresponding mode shapes proposed by Lee and Jung [7, 8] is extended to the proportionally and non-proportionally damped systems with multiple natural frequencies. In the case of multiple eigenvalues as well as distinct ones, the proposed method can find the mode shape derivatives by solving the algebraic equation with *symmetric coefficient matrix added side conditions*. The orthonormal condition and a set of adjacent eigenvectors can be used in the algebraic equation as side conditions.

The second section of this paper presents the proposed sensitivity analysis method of damped systems with multiple natural frequencies. The third section presents numerical stability of the proposed method, and the next section numerical examples.

2. SENSITIVITY ANALYSIS OF A DAMPED SYSTEM WITH MULTIPLE NATURAL FREQUENCIES

When a natural frequency has multiplicity m and a design parameter is perturbed, the corresponding mode shapes may split into as many as m distinct mode shapes. For derivatives of the mode shapes to be responsible, the mode shapes must be laid adjacent to the m distinct mode shapes that appear when a design parameter varies. Otherwise, the mode shapes would jump discontinuously with a varying design parameter. Here the derivatives of these adjacent mode shapes are sought.

The eigenvalue problem of a damped system can be expressed as

$$(\lambda^2\mathbf{M} + \lambda\mathbf{C} + \mathbf{K})\phi = 0, \tag{1}$$

where $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are the matrices of mass, damping and stiffness, respectively, and these are order n symmetric matrices. $\mathbf{M}$ is positive definite and $\mathbf{K}$ positive definite or semi-positive definite. The first step in finding derivatives of mode shapes of multiple eigenvalues is to find corresponding adjacent mode shapes. Suppose that all eigenpairs are known and multiplicity of the eigenvalue λ_m is m. Define the following eigenvalue problem where $\mathbf{\Phi}_m$ is the matrix of eigenvectors corresponded to the multiple eigenvalue, hence, its order $(n \times m)$.

$$\mathbf{M}\mathbf{\Phi}_m\Lambda_m^2 + \mathbf{C}\mathbf{\Phi}_m\Lambda_m + \mathbf{K}\mathbf{\Phi}_m = 0, \tag{2}$$

where

$$\Lambda_m = \lambda_m \mathbf{I}_m \quad \text{and} \quad \mathbf{\Phi}_m = [\phi_{i+1}\phi_{i+2}\ldots\phi_{i+m}]. \tag{3}$$

$\mathbf{I}_m$ is the identity matrix of order m and λ_m is the eigenvalue of multiplicity m for the eigenspace spanned by the columns of $\mathbf{\Phi}_m$. As noted in Part I, the orthonormal condition for the $(i+1)$th eigenvector is as follows:

$$\phi_{i+1}^T(2\lambda_{i+1}\mathbf{M} + \mathbf{C})\phi_{i+1} = 1. \tag{4}$$

Since the multiplicity is m, the orthonormal condition for the matrix $\mathbf{\Phi}_m$ is as follows:

$$\mathbf{\Phi}_m^T(2\lambda_m\mathbf{M} + \mathbf{C})\mathbf{\Phi}_m = \mathbf{I}_m. \tag{5}$$

Adjacent eigenvectors can be expressed in terms of $\mathbf{\Phi}_m$ by an orthogonal transformation such as

$$\mathbf{X} = \mathbf{\Phi}_m\mathbf{T}, \tag{6}$$

where $\mathbf{T}$ is an orthonormal transformation matrix and its order m;

$$\mathbf{T}^T\mathbf{T} = \mathbf{I}_m. \tag{7}$$

The columns of $\mathbf{X}$ are the adjacent eigenvectors for which a derivative can be defined. It is natural that the adjacent eigenvectors satisfy the orthonormal condition too:

$$\mathbf{X}^T(2\lambda_m\mathbf{M} + \mathbf{C})\mathbf{X} = \mathbf{T}^T\mathbf{\Phi}_m^T(2\lambda_m\mathbf{M} + \mathbf{C})\mathbf{\Phi}_m\mathbf{T} = \mathbf{T}^T\mathbf{T} = \mathbf{I}_m. \tag{8}$$

The next procedure is to find $\mathbf{T}$ and then to find $\mathbf{X}$ and $\partial\Lambda_m/\partial p$. If design parameter p varies, $\partial\Lambda_m/\partial p$ is expressed as

$$\frac{\partial\mathbf{\Lambda}_m}{\partial p} = \operatorname{diag}\left(\frac{\partial\lambda_{i+1}}{\partial p},\ \frac{\partial\lambda_{i+2}}{\partial p},\ \ldots,\ \frac{\partial\lambda_{i+m}}{\partial p}\right). \tag{9}$$

Consider another eigenvalue problem to find X and $\partial\Lambda_m/\partial p$.

$$\mathbf{M}\mathbf{X}\mathbf{\Lambda}_m^2 + \mathbf{C}\mathbf{X}\mathbf{\Lambda}_m + \mathbf{K}\mathbf{X} = 0, \tag{10}$$

where the order of adjacent eigenvector matrix $\mathbf{X}$ is $(n\times m)$ and the order of eigenvalue matrix $\mathbf{\Lambda}_m$ is $(m\times m)$. Differentiating the above eigenvalue problem with respect to the design parameter p, and rearranging yields

$$[\lambda_m^2\mathbf{M} + \lambda_m\mathbf{C} + \mathbf{K}]\frac{\partial\mathbf{X}}{\partial p} = -(2\lambda_m\mathbf{M} + \mathbf{C})\mathbf{X}\frac{\partial\mathbf{\Lambda}_m}{\partial p} - \left(\lambda_m^2\frac{\partial\mathbf{M}}{\partial p} + \lambda_m\frac{\partial\mathbf{C}}{\partial p} + \frac{\partial\mathbf{K}}{\partial p}\right)\mathbf{X}. \tag{11}$$

Premultiplying each side of equation (11) by $\mathbf{\Phi}_m^T$ and substituting $\mathbf{X} = \mathbf{\Phi}_m\mathbf{T}$ into it gives a new eigenvalue problem such as

$$\mathbf{D}\mathbf{T} = \mathbf{E}\mathbf{T}\frac{\partial\mathbf{\Lambda}_m}{\partial p}, \tag{12}$$

where

$$\mathbf{D} = \mathbf{\Phi}_m^T \left(\lambda_m^2 \frac{\partial \mathbf{M}}{\partial p} + \lambda_m \frac{\partial \mathbf{C}}{\partial p} + \frac{\partial \mathbf{K}}{\partial p} \right) \mathbf{\Phi}_m \quad \text{and} \quad \mathbf{E} = -\mathbf{\Phi}_m^T (2\lambda_m \mathbf{M} + \mathbf{C}) \mathbf{\Phi}_m = -\mathbf{I}_m. \tag{13}$$

One can obtain the eigenvalue derivative $\partial \mathbf{\Lambda}_m / \partial p$ and orthogonal transformation matrix $\mathbf{T}$ by solving equation (12), and then the adjacent eigenvectors by relation $\mathbf{X} = \mathbf{\Phi}_m \mathbf{T}$. A similar procedure for finding the adjacent eigenvectors in the case of the undamped system is derived by Chen and Pan [10].

The proposed method starts with the equations of the derivative of the eigenvalue problem composed of the system matrices and the adjacent eigenvectors, equation (11), and the orthonormal condition, equation (8). Differentiating equation (8) with respect to the design parameter gives

$$\mathbf{X}^T (2\lambda_m \mathbf{M} + \mathbf{C}) \frac{\partial \mathbf{X}}{\partial p} = -\mathbf{X}^T \frac{\partial \mathbf{M}}{\partial p} \mathbf{X} \mathbf{\Lambda}_m - \mathbf{X}^T \mathbf{M} \mathbf{X} \frac{\partial \mathbf{\Lambda}_m}{\partial p} - \frac{1}{2} \mathbf{X}^T \frac{\partial \mathbf{C}}{\partial p} \mathbf{X}. \tag{14}$$

One can write the following single matrix equation by combining equations (11) and (14).

$$\begin{bmatrix} \lambda_m^2 \mathbf{M} + \lambda_m \mathbf{C} + \mathbf{K} & (2\lambda_m \mathbf{M} + \mathbf{C}) \mathbf{X} \\ \mathbf{X}^T (2\lambda_m \mathbf{M} + \mathbf{C}) & \mathbf{0} \end{bmatrix} \begin{bmatrix} \dfrac{\partial \mathbf{X}}{\partial p} \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} -(2\lambda_m \mathbf{M} + \mathbf{C}) \mathbf{X} \dfrac{\partial \mathbf{\Lambda}_m}{\partial p} - \left(\lambda_m^2 \dfrac{\partial \mathbf{M}}{\partial p} + \lambda_m \dfrac{\partial \mathbf{C}}{\partial p} + \dfrac{\partial \mathbf{K}}{\partial p} \right) \mathbf{X} \\ -\mathbf{X}^T \dfrac{\partial \mathbf{M}}{\partial p} \mathbf{X} \mathbf{\Lambda}_m - \mathbf{X}^T \mathbf{M} \mathbf{X} \dfrac{\partial \mathbf{\Lambda}_m}{\partial p} - \dfrac{1}{2} \mathbf{X}^T \dfrac{\partial \mathbf{C}}{\partial p} \mathbf{X} \end{bmatrix}, \tag{15}$$

where the order of coefficient matrix on the left side of equation (15) is $(n+m) \times (n+m)$ and the matrix on the right side is $(n+m) \times m$. The derivatives $\partial \mathbf{X} / \partial p$ can be found by solving equation (15). The coefficient matrix can be decomposed into upper and lower triangular forms [11] and then a forward and backward substitution scheme may be used to evaluate the components of $\partial \mathbf{X} / \partial p$.

Note that the proposed method has the desirable properties of preserving the structure of the system matrices, and of requiring knowledge of only multiple eigenpairs. Note also that the proposed method needs the first order derivatives of the mass, damping and stiffness matrices, whereas Dailey's method which finds the exact solutions needs both the first and second derivatives. The numerical stability of the proposed algorithm for multiple eigenvalues will be proved in section 3. The procedures for the sensitivity analysis method in the case of multiple eigenvalues are summarized in Table 1.

TABLE 1
The procedure of the proposed method

(1) Compute $\mathbf{D} = \mathbf{\Psi}^T\left(\lambda_m^2\frac{\partial \mathbf{M}}{\partial p} + \lambda_m\frac{\partial \mathbf{C}}{\partial p} + \frac{\partial \mathbf{K}}{\partial p}\right)\mathbf{\Psi}$ and $\mathbf{E} = -\mathbf{I}_m$.

(2) Solve the eigenvalue problem $\mathbf{DT} = \mathbf{ET}\frac{\partial \mathbf{\Lambda}_m}{\partial p}$ and normalize so that $\mathbf{T}^T\mathbf{T} = \mathbf{I}_m$.

(3) Let the columns of $\mathbf{X} = \mathbf{\Psi T}$ be the new eigenvectors.

(4) Define $\mathbf{A}^* = \begin{bmatrix} \lambda_m^2\mathbf{M} + \lambda_m\mathbf{C} + \mathbf{K} & (2\lambda_m\mathbf{M} + \mathbf{C})\mathbf{X} \\ \mathbf{X}^T(2\lambda_m\mathbf{M} + \mathbf{C}) & \mathbf{0} \end{bmatrix}$.

(5) Compute $F = \begin{Bmatrix} -(2\lambda_m\mathbf{M} + \mathbf{C})\mathbf{X}\frac{\partial \mathbf{\Lambda}_m}{\partial p} - \left(\lambda_m^2\frac{\partial \mathbf{M}}{\partial p} + \lambda_m\frac{\partial \mathbf{C}}{\partial p} + \frac{\partial \mathbf{K}}{\partial p}\right)\mathbf{X} \\ -\mathbf{X}^T\frac{\partial \mathbf{M}}{\partial p}\mathbf{X}\mathbf{\Lambda}_m - \mathbf{X}^T\mathbf{M}\mathbf{X}\frac{\partial \mathbf{\Lambda}_m}{\partial p} - \frac{1}{2}\mathbf{X}^T\frac{\partial \mathbf{C}}{\partial p}\mathbf{X} \end{Bmatrix}$.

(6) Compute $\begin{bmatrix} \frac{\partial \mathbf{X}}{\partial p} \\ 0 \end{bmatrix} = [\mathbf{A}^*]^{-1}\mathbf{F}$.

3. NUMERICAL STABILITY OF THE PROPOSED METHOD

Identifying the nonsingularity of the coefficient matrix $\mathbf{A}^*$ of order $(n+m)$ in equations (15) and (16) may be used to prove the numerical stability of the proposed method in the case of multiple eigenvalues.

$$\mathbf{A}^* = \begin{bmatrix} \lambda_m^2\mathbf{M} + \lambda_m\mathbf{C} + \mathbf{K} & (2\lambda_m\mathbf{M} + \mathbf{C})\mathbf{X} \\ \mathbf{X}^T(2\lambda_m\mathbf{M} + \mathbf{C}) & \mathbf{0} \end{bmatrix}, \tag{16}$$

where $\mathbf{X}$ is a $n \times m$ matrix having adjacent eigenvectors as its columns, and λ_m is the multiple eigenvalue of multiplicity m. To show that the coefficient matrix $\mathbf{A}^*$ is always non-singular, consider another matrix such as $\mathbf{Y}^T\mathbf{A}^*\mathbf{Y}$ where $\mathbf{Y}$ is a $(n+m)\times(n+m)$ non-singular matrix. If it is proved that the determinant of $\mathbf{Y}^T\mathbf{A}^*\mathbf{Y}$ is non-zero, then the determinant of matrix $\mathbf{A}^*$ may also be non-zero and $\mathbf{A}^*$ is non-singular.

In this paper, the matrix $\mathbf{Y}$ is assumed as

$$\mathbf{Y} = \begin{bmatrix} \mathbf{\Psi} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_m \end{bmatrix}, \tag{17}$$

where $\mathbf{I}_m$ is an identity matrix of order m and $\mathbf{\Psi}$ is a set of arbitrary independent vectors containing the adjacent eigenvectors of multiple eigenvalue λ_m of the

system, as follows

$$\boldsymbol{\Psi} = [\psi_1 \quad \psi_2 \quad \ldots \quad \psi_{n-m} \quad \mathbf{x}_1 \quad \mathbf{x}_2 \quad \ldots \quad \mathbf{x}_m] \quad \text{when} \quad \mathbf{X} = [\mathbf{x}_1 \quad \mathbf{x}_2 \quad \ldots \quad \mathbf{x}_m], \tag{18}$$

where ψ's are arbitrary independent vectors chosen to be independent to the adjacent eigenvector $\mathbf{x}$'s. Since all the columns of the matrix $\mathbf{Y}$ are independent vectors, matrix $\mathbf{Y}$ is non-singular and so it is invertible. Pre- and post-multiplying $\mathbf{Y}^T$ and $\mathbf{Y}$ to $\mathbf{A}^*$ yields

$$\mathbf{Y}^T\mathbf{A}^*\mathbf{Y} = \begin{bmatrix} \boldsymbol{\Psi} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_m \end{bmatrix}^T \left[\begin{array}{c|c} \lambda_m^2\mathbf{M} + \lambda_m\mathbf{C} + \mathbf{K} & (2\lambda_m\mathbf{M} + \mathbf{C})\mathbf{X} \\ \hline \mathbf{X}^T(2\lambda_m\mathbf{M} + \mathbf{C}) & \mathbf{0} \end{array}\right] \begin{bmatrix} \boldsymbol{\Psi} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_m \end{bmatrix}$$

$$= \left[\begin{array}{c|c} \boldsymbol{\Psi}^T(\lambda_m^2\mathbf{M} + \lambda_m\mathbf{C} + \mathbf{K})\boldsymbol{\Psi} & \boldsymbol{\Psi}^T(2\lambda_m\mathbf{M} + \mathbf{C})\mathbf{X} \\ \hline \mathbf{X}^T(2\lambda_m\mathbf{M} + \mathbf{C})\boldsymbol{\Psi} & \mathbf{0} \end{array}\right]. \tag{19}$$

It is obvious that the last m columns and rows of the matrix $\boldsymbol{\Psi}^T(\lambda_m^2\mathbf{M} + \lambda_m\mathbf{C} + \mathbf{K})\boldsymbol{\Psi}$ all have zero elements, which are provided by the eigenvalue problem $(\lambda_m^2\mathbf{M} + \lambda_m\mathbf{C} + \mathbf{K})\mathbf{X} = \mathbf{0}$, as follows

$$\boldsymbol{\Psi}^T(\lambda_m^2\mathbf{M} + \lambda_m\mathbf{C} + \mathbf{K})\boldsymbol{\Psi} = \begin{bmatrix} \tilde{\mathbf{A}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \tag{20}$$

where $\tilde{\mathbf{A}}$ is a non-zero $(n-m)\times(n-m)$ submatrix. The submatrix $\tilde{\mathbf{A}}$ is a non-singular matrix having order of $n-m$ and rank of $n-m$, since it is given by eliminating the columns and rows having all zero elements from $\boldsymbol{\Psi}^T(\lambda_m^2\mathbf{M} + \lambda_m\mathbf{C} + \mathbf{K})\Omega$ of order n and rank $n-m$. That is, $\det(\tilde{\mathbf{A}}) \neq 0$.

By the normalization condition,

$$\boldsymbol{\Psi}^T(2\lambda_m\mathbf{M} + \mathbf{C})\mathbf{X} = \begin{Bmatrix} \tilde{\mathbf{B}} \\ \mathbf{I}_m \end{Bmatrix} \quad \text{and} \quad \mathbf{X}^T(2\lambda_m\mathbf{M} + \mathbf{C})\boldsymbol{\Psi} = \begin{Bmatrix} \tilde{\mathbf{B}} \\ \mathbf{I}_m \end{Bmatrix}, \tag{21}$$

where $\tilde{\mathbf{B}}$ is generally a non-zero rectangular matrix. Substituting equations (20) and (21), into equation (19) yields

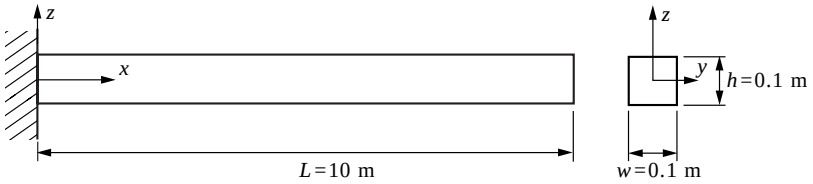

Figure 1. Cantilever beam with height h as the design parameter. Number of nodes: 21; number of elements: 20; number of degrees of freedom: 80; Young's modulus: $E = 2{\cdot}10 \times 10^{11}$; mass density: $p = 7{\cdot}85 \times 10^3$ kg/m^3.

$$\mathbf{Y}^T\mathbf{A}^*\mathbf{Y} = \begin{bmatrix} \tilde{\mathbf{A}} & \mathbf{0} & \tilde{\mathbf{B}} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_m \\ \tilde{\mathbf{B}}^T & \mathbf{I}_m & \mathbf{0} \end{bmatrix}. \tag{22}$$

By applying the matrix determinant property of partitioned matrices, the determinant of $\mathbf{Y}^T\mathbf{A}^*\mathbf{Y}$ can be rewritten as

$$\det(\mathbf{Y}^T\mathbf{A}^*\mathbf{Y}) = \det\begin{bmatrix} \mathbf{0} & \mathbf{I}_m \\ \mathbf{I}_m & \mathbf{0} \end{bmatrix} \det\left(\tilde{\mathbf{A}} - [\mathbf{0} \quad \tilde{\mathbf{B}}] \begin{bmatrix} \mathbf{0} & \mathbf{I}_m \\ \mathbf{I}_m & \mathbf{0} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{0} \\ \tilde{\mathbf{B}}^T \end{bmatrix} \right) \tag{23}$$

or

$$\det(\mathbf{Y}^T\mathbf{A}^*\mathbf{Y}) = \det(\tilde{\mathbf{A}}) \neq 0. \tag{24}$$

The determinant of $\mathbf{A}^*$ thus is not equal to zero because $\det(\mathbf{Y}^T\mathbf{A}^*\mathbf{Y}) \neq 0$. The proof is completed mathematically for the numerical stability of the proposed algorithm in the case of multiple eigenvalues.

4. NUMERICAL EXAMPLES

To demonstrate the theory of the proposed method and its possibilities in the case of multiple natural frequencies, two examples are presented. The first example is the finite element model of a cantilever beam as the proportionally damped system. The second example is a 5-DOF mechanical system as the non-proportionally damped system.

4.1. CANTILEVER BEAM (PROPORTIONALLY DAMPED SYSTEM)

As an illustrative example in the case of the proportionally damped system with multiple natural frequencies, the cantilever beam with square section used in reference [12] is considered. Each member is modelled as a beam element of which each node has four degrees of freedom (y-translation, z-translation, y-rotation and z-rotation), as shown in Figure 1. The number of nodes is 21, and the number of elements 20. Each element has eight degrees of freedom, and the structure has 80 degrees of freedom. Young's modulus is $2{\cdot}10 \times 10^{11}$ N/m^2 and mass density $7{\cdot}85 \times 10^3$ kg/m^3. Both the beam height and width are 0·1 m, and its length 10 m.

Assume that the damping matrix is a linear combination of the stiffness and mass matrices as

$$\mathbf{C} = \alpha\mathbf{K} + \beta\mathbf{M} \tag{25}$$

where α and β are the Rayleigh coefficients. The design parameter is the beam height h.

Some results are shown in Table 2. The lowest 20 eigenvalues and their derivatives of the cantilever beam are listed in the second and third columns of Table 2. The first and second eigenvalue conjugate pairs are the same, the third

TABLE 2

The lowest 20 *eigenvalues of the initial and changed cantilever beam system, and results of the sensitivity analysis*

Mode number	Initial system		Changed system		Variation of eigenpair		Error of approximation	
	Eigenvalue	Eigenvalue derivative	Eigenvalue	Approximated eigenvalue	Eigenvalue	Eigenvector	Eigenvalue	Eigenvector
1, 2	$-1{\cdot}4279\times10^{-3}$ $\pm j5{\cdot}2496\times10^{0}$	$9{\cdot}4043\times10^{-10}$ $\mp j2{\cdot}1262\times10^{-10}$	$-1{\cdot}4279\times10^{-3}$ $\pm j5{\cdot}2496\times10^{0}$	$-1{\cdot}4279\times10^{-3}$ $\pm j5{\cdot}2496\times10^{0}$	$3{\cdot}4421\times10^{-11}$	$4{\cdot}9628\times10^{-3}$	$3{\cdot}4381\times10^{-11}$	$3{\cdot}7376\times10^{-5}$
3, 4	$-1{\cdot}4279\times10^{-3}$ $\pm j5{\cdot}2496\times10^{0}$	$-2{\cdot}7557\times10^{-2}$ $\pm j5{\cdot}2494\times10^{1}$	$-1{\cdot}4556\times10^{-3}$ $\pm j5{\cdot}3021\times10^{0}$	$-1{\cdot}4555\times10^{-3}$ $\pm j5{\cdot}3021\times10^{0}$	$9{\cdot}9996\times10^{-3}$	$9{\cdot}9012\times10^{-3}$	$2{\cdot}6631\times10^{-8}$	$1{\cdot}0000\times10^{-4}$
5, 6	$-5{\cdot}4154\times10^{-2}$ $\pm j3{\cdot}2895\times10^{1}$	$-1{\cdot}9577\times10^{-9}$ $\mp 9{\cdot}1389\times10^{-10}$	$-5{\cdot}4154\times10^{-2}$ $3{\cdot}2895\times10^{1}$	$-5{\cdot}4154\times10^{-2}$ $\pm j3{\cdot}2895\times10^{1}$	$7{\cdot}0587\times10^{-12}$	$4{\cdot}9628\times10^{-3}$	$6{\cdot}9949\times10^{-12}$	$3{\cdot}7376\times10^{-5}$
7, 8	$-5{\cdot}4154\times10^{-2}$ $\pm j3{\cdot}2895\times10^{1}$	$-1{\cdot}0818\times10^{0}$ $\pm j3{\cdot}2886\times10^{2}$	$-5{\cdot}5241\times10^{-2}$ $\pm j3{\cdot}3224\times10^{1}$	$-5{\cdot}5236\times10^{-2}$ $3{\cdot}3224\times10^{1}$	$9{\cdot}9973\times10^{-3}$	$9{\cdot}9023\times10^{-3}$	$1{\cdot}6763\times10^{-7}$	$1{\cdot}0001\times10^{-4}$
9, 10	$-4{\cdot}2409\times10^{-1}$ $\pm 9{\cdot}2090\times10^{1}$	$8{\cdot}6743\times10^{-10}$ $\mp j4{\cdot}1110\times10^{-9}$	$-4{\cdot}2409\times10^{-1}$ $\pm j9{\cdot}2090\times10^{1}$	$-4{\cdot}2409\times10^{-1}$ $\pm j9{\cdot}2090\times10^{1}$	$7{\cdot}0586\times10^{-12}$	$4{\cdot}9628\times10^{-3}$	$7{\cdot}0132\times10^{-12}$	$3{\cdot}7376\times10^{-5}$
11, 12	$-4{\cdot}2409\times10^{-1}$ $\pm j9{\cdot}2090\times10^{1}$	$-8{\cdot}4753\times10^{-0}$ $\pm j9{\cdot}2029\times10^{2}$	$-4{\cdot}3260\times10^{-1}$ $\pm j9{\cdot}3010\times10^{1}$	$-4{\cdot}3256\times10^{-1}$ $\pm j9{\cdot}3010\times10^{1}$	$9{\cdot}9936\times10^{-3}$	$9{\cdot}9041\times10^{-3}$	$4{\cdot}6508\times10^{-7}$	$1{\cdot}0002\times10^{-4}$
13, 14	$-1{\cdot}6276\times10^{0}$ $\pm j1{\cdot}8041\times10^{2}$	$1{\cdot}6719\times10^{-9}$ $\mp j3{\cdot}8254\times10^{-9}$	$-1{\cdot}6276\times10^{0}$ $\pm j1{\cdot}8041\times10^{2}$	$-1{\cdot}6276\times10^{0}$ $\pm j1{\cdot}8041\times10^{2}$	$4{\cdot}9435\times10^{-12}$	$4{\cdot}9628\times10^{-3}$	$4{\cdot}9204\times10^{-12}$	$3{\cdot}7376\times10^{-5}$
15, 16	$-1{\cdot}6276\times10^{0}$ $\pm j1{\cdot}8041\times10^{2}$	$-3{\cdot}2513\times10^{1}$ $\pm j1{\cdot}8018\times10^{3}$	$-1{\cdot}6603\times10^{0}$ $\pm j1{\cdot}8222\times10^{2}$	$-1{\cdot}6601\times10^{0}$ $\pm j1{\cdot}8222\times10^{2}$	$9{\cdot}9883\times10^{-3}$	$9{\cdot}9064\times10^{-3}$	$9{\cdot}0775\times10^{-7}$	$1{\cdot}0003\times10^{-4}$
17, 18	$-4{\cdot}4454\times10^{0}$ $\pm j2{\cdot}9814\times10^{2}$	$-1{\cdot}0635\times10^{-9}$ $\pm j1{\cdot}0525\times10^{-8}$	$-4{\cdot}4454\times10^{0}$ $\pm j2{\cdot}9814\times10^{2}$	$-4{\cdot}4454\times10^{0}$ $\pm j2{\cdot}9814\times10^{2}$	$3{\cdot}7901\times10^{-13}$	$4{\cdot}9628\times10^{-3}$	$3{\cdot}5735\times10^{-13}$	$3{\cdot}7376\times10^{-5}$
19, 20	$-4{\cdot}4454\times10^{0}$ $\pm j2{\cdot}9814\times10^{2}$	$-8{\cdot}8739\times10^{1}$ $\pm j2{\cdot}9751\times10^{23}$	$-4{\cdot}5346\times10^{0}$ $\pm j3{\cdot}0112\times10^{2}$	$-4{\cdot}5342\times10^{0}$ $\pm j3{\cdot}0112\times10^{2}$	$9{\cdot}9819\times10^{-3}$	$9{\cdot}9090\times10^{-3}$	$1{\cdot}4956\times10^{-6}$	$1{\cdot}0004\times10^{-4}$

and fourth, and so on. One can see that the derivatives of the multiple eigenvalues are different in that one is close to zero while the other is not. Since the design parameter is the height of the beam h, when h is varied, the multiple eigenvalues are split into distinct ones as the cross-section of the beam is no longer square after changing the height. To illustrate the sensitivity analysis results, the actual and approximate values of the changed system of $\Delta h/h = 0{\cdot}01$ are represented in the fourth and fifth columns of Table 2. The next two columns are the variation of eigenvalues and eigenvectors between initial and changed ones and the last two are errors of the approximations. The errors are reasonably smaller than the corresponding variations, and one can say that the proposed method gives good results for the case of multiple eigenvalues and for a proportionally damped system.

4.2. PRIMARY AND SECONDARY SYSTEMS EQUIPPED ON THE RIGID SQUARE PLATE (NON-PROPORTIONALLY DAMPED SYSTEM)

An analytical example to verify the proposed method in the case of the non-proportionally damped system with multiple eigenvalues, the 5-DOF mass, spring and damper system shown in Figure 2 is considered. Assume that only vibrations in the vertical plane are possible.

The components of the mass matrix $\mathbf{M}$ of the system m_{ij}'s are:

$$m_{11} = m_1, \quad m_{23} = m_2, \quad m_{33} = m_3, \quad m_{44} = J_4, \quad m_{55} = J_5,$$
$$\text{and} \quad m_{ij} = 0 \quad \text{if} \quad i \neq j$$

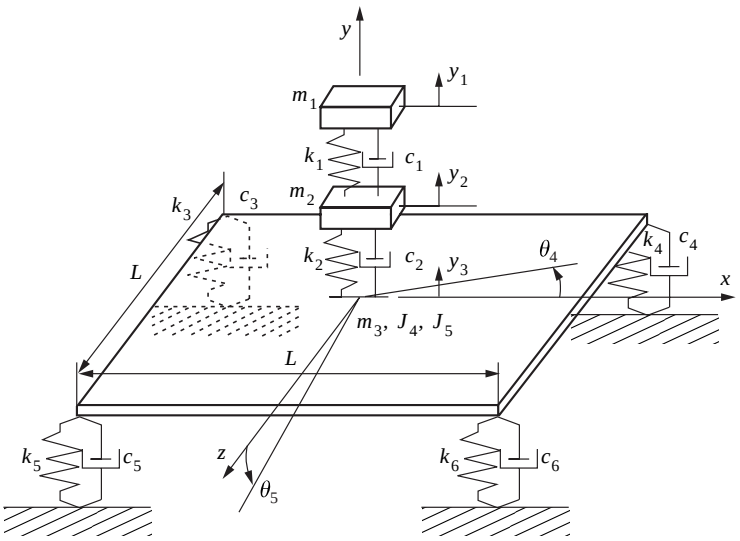

Figure 2. 5-DOF non-proportionally damped system. $m_1 = 200$ kg, $m_2 = 500$ kg, $m_3 = 1000$ kg, $k_1 = 10\,000$ N/m, $k_2 = 20\,000$ N/m, $k_3 = k_4 = k_5 = k_6 = 1000$ N/m, $c_1 = 4$ Ns/m, $c_2 = 6$ Ns/m, $c_3 = c_4 = c_5 = c_6 = 40$ Ns/m.

The elements of the stiffness matrix k_{ij}'s are given as:

$$k_{11} = k_1; \quad k_{12} = -k_1; \quad k_{13} = k_{14} = k_{15} = 0;$$

$$k_{22} = k_1 + k_2; \quad k_{23} = -k_2; \quad k_{24} = k_{25} = 0;$$

$$k_{33} = k_2 + k_3 + k_4 + k_5 + k_6; \quad k_{34} = -L/2(k_3 - k_4 + k_5 - k_6);$$

$$k_{35} = -L/2(k_3 + k_4 - k_5 - k_6); \quad k_{44} = (L/2)^2(k_3 + k_4 + k_5 + k_6);$$

$$k_{45} = (L/2)^2(k_3 - k_4 - k_5 - k_6); \quad k_{55} = (L/2)^2(k_3 + k_4 + k_5 + k_6).$$

The damping matrix **C** has an analogous form to the stiffness matrix: e.g.,

$$c_{11} = c_1; \quad c_{12} = -c_1; \quad c_{13} = c_{14} = c_{15} = 0; \quad c_{22} = c_1 + c_2, \quad \text{etc.}$$

Some results are summarized in Table 3 which shows the eigenvalues of the system and their sensitivities when the design parameter is k_5. Note that the second and third eigenvalue conjugate pairs are multiples respectively. The derivatives of the multiple eigenvalues are different since the design parameter is the spring k_5; when k_5 is varied, the multiple eigenvalues are split into distinct ones since the structural symmetry is broken. The exact and approximated eigenvalues of the system after changing k_5 by $\Delta k_5/k_5 = 0{\cdot}01$ are represented in the fourth and fifth columns of the table. The last four columns are variations of exact eigenpairs and errors of the approximate eigenpairs. Since the sensitivities of the second and third eigenpairs are equal to zero, the second and third eigenpairs are not changed. Considering that the errors of the approximate eigenpairs are relatively smaller than the variations, the approximate eigenvalues and eigenvectors of the changed system are reasonable. Consequently, one can say that the proposed method gives good results.

The proposed method is verified through examples. The proposed method can be applied very well to the proportionally and non-proportionally damped systems and to the eigenvalue problem with multiple eigenvalues as well as distinct ones (refer to Part I).

5. CONCLUSIONS

This paper proposes an efficient numerical method for calculating vibration mode shape derivatives of the proportionally and non-proportionally damped systems with multiple eigenvalues. The method finds eigenpair derivatives of the systems by solving the linear algebraic equation without any numerical instability. The proposed method is very efficient in the case of the multiple eigenvalue problems since the computer storage and analysis time required are smaller than those of Dailey's method, since our method does not use second derivatives of the system matrices while Dailey's method does. The proposed method may be inserted easily into a commercial FEM code since it finds the exact solution and treats a symmetric matrix. Furthermore, its algorithm is very simple and numerically stable.

TABLE 3

The natural frequencies of the initial and changed primary and secondary system, and results of the sensitivity analysis

	Initial system		Changed system		Variation of eigenpair		Error of approximation	
Mode number	Eigenvalue	Eigenvalue derivative	Eigenvalue	Approximated eigenvalue	Eigenvalue	Eigenvector	Eigenvalue	Eigenvector
1, 2	$-4{\cdot}3262 \times 10^{-2}$ $\pm j1{\cdot}5023 \times 10^{0}$	$9{\cdot}6943 \times 10^{-7}$ $\pm j1{\cdot}7995 \times 10^{-4}$	$-4{\cdot}3253 \times 10^{-2}$ $\pm j1{\cdot}5040 \times 10^{0}$	$-4{\cdot}3253 \times 10^{-2}$ $\pm j1{\cdot}5041 \times 10^{0}$	$1{\cdot}1893 \times 10^{-3}$	$4{\cdot}6721 \times 10^{-3}$	$8{\cdot}1631 \times 10^{-6}$	$2{\cdot}9463 \times 10^{-5}$
3, 4	$-2{\cdot}4000 \times 10^{-1}$ $\pm j3{\cdot}4558 \times 10^{0}$	$0{\cdot}0000 \times 10^{0}$ $\pm j0{\cdot}0000 \times 10^{0}$	$-2{\cdot}4000 \times 10^{-1}$ $\pm j3{\cdot}4558 \times 10_{0}$	$-2{\cdot}4000 \times 10^{-1}$ $-2{\cdot}4000 \times 10^{01}$	$0{\cdot}0000 \times 10^{0}$	$0{\cdot}0000 \times 10^{0}$	$0{\cdot}0000 \times 10^{0}$	$0{\cdot}0000 \times 10^{0}$
5, 6	$-2{\cdot}4000 \times 10^{-1}$ $\pm j3{\cdot}4558 \times 10^{0}$	$-0{\cdot}0000 \times 10^{0}$ $\pm 8{\cdot}6811 \times 10^{-4}$	$-2{\cdot}4000 \times 10^{-1}$ $\pm j3{\cdot}4645 \times 10^{0}$	$-2{\cdot}4000 \times 10^{-1}$ $\pm j3{\cdot}4645 \times 10^{0}$	$2{\cdot}5039 \times 10^{-3}$	$1{\cdot}5461 \times 10^{-3}$	$2{\cdot}1632 \times 10^{-6}$	$5{\cdot}2014 \times 10^{-6}$
7, 8	$-3{\cdot}5202 \times 10^{-2}$ $\pm j6{\cdot}1354 \times 10^{0}$	$-7{\cdot}8926 \times 10^{-7}$ $\pm j2{\cdot}9526 \times 10^{-5}$	$-3{\cdot}5210 \times 10^{-2}$ $\pm j6{\cdot}1357 \times 10^{0}$	$-3{\cdot}5210 \times 10^{-2}$ $\pm j6{\cdot}1357 \times 10^{0}$	$4{\cdot}8257 \times 10^{-5}$	$1{\cdot}0987 \times 10^{-3}$	$1{\cdot}1763 \times 10^{-7}$	$2{\cdot}5394 \times 10^{-6}$
9, 10	$-2{\cdot}4535 \times 10^{-2}$ $\pm 9{\cdot}7000 \times 10^{0}$	$-1{\cdot}8017 \times 10^{-7}$ $\pm j5{\cdot}0001 \times 10^{-6}$	$-2{\cdot}4537 \times 10^{-2}$ $\pm j9{\cdot}7000 \times 10^{0}$	$-2{\cdot}4537 \times 10^{-2}$ $\pm j9{\cdot}7000 \times 10^{0}$	$5{\cdot}1624 \times 10^{-6}$	$1{\cdot}9442 \times 10^{-4}$	$4{\cdot}3893 \times 10^{-9}$	$1{\cdot}6332 \times 10^{-7}$

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